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FINAL REPORT

INVESTIGATION OF THE ORBITAL TRACKING OF A GEOSTATIONARY SATELLITE

VOLUME I OF II
THEORY AND METHODS

12 February 1962

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PREPARED FOR

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Astrosciences Department

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VOLUME I

THEORY AND METHODS

CONTENTS

SECTION 1

INTRODUCTION

This summary report conveys the results of a seven-month contract on The Investigation of the Orbital Tracking of a Geostationary Satellite. One purpose of the study is to determine the effects of uncertainties in observation quantities on the orbit parameters as they are propagated in the orbit correction technique. This determination considers various tracking concepts, accuracies, frequencies, and tracking deviations. A second purpose is to determine the effect of bias errors on the orbit determination method and, depending on the magnitude of the effect, ascertain schemes for treating these biases.

The report is presented in two volumes of which this, Volume I, concerns itself with the theory and methods used in the study. Volume II presents the numerical and graphical results derived from the investigation.

In Volume I will be found a discussion of the statistical techniques used in this study. The method of least squares is the mathematical tool employed in the analysis under the assumption that the uncertainties in the observations are randomly distributed. The effect of noise correlation between observations of the same type is established in order to define an upper bound on sampling frequency in the investigation.

The orbit is defined by the $\underline{U}$, $\underline{V}$ set of parameters which circumvent the usual singularities associated with zero inclination and zero eccentricity. Analytical differential expressions between these parameters and the observed quantities of range, range rate, right ascension and declination are developed for use in the least squares error analysis, which produces the statistical variances in the orbital elements. These variances are combined to form the variances in the following mission parameters which were established in the initial stages of the study:

(1) Time to drift out of a 2° topocentric cone

(2) North-south excursion

(3) East-west excursion

(4) East-west bias.

The final sections of Volume I are concerned with the subject of bias errors. Here the biases in station location, uncertainties in gravitational and celestial constants, uncertainty in the velocity of propagation, and atmospheric refraction correction uncertainties are enumerated. Analytic expressions are developed relating these bias errors to the observed quantities, and in instances where possible they are related directly to uncertainties in specific orbit parameters. Matrix methods are developed for the treatment of biases providing a technique for extending the matrix of partials to include the bias error partials. A similar matrix development is applied to the differential correction process, thereby providing a method of detecting the magnitude of a bias and subsequently correcting it.

SECTION 2

METHOD OF ERROR ANALYSIS

This study deals with the relationships between tracking measurements and the quality of the orbit derived therefrom. Considered here is the problem of predicting the position of a geostationary satellite based on information from instrumentation of a given reliability. Errors in the prediction are then compared with some acceptable limit, where errors in the measurements are assumed to be of a random nature and statistically correlated.

2.1 LEAST SQUARES ERROR ANALYSIS

Any observation 0_i by an earth-fixed observer may be expressed in terms of six parameters or "elements" describing the orbit and the time. First order differential expressions relating observation and parameters follow from the leading term in a Taylor expansion, i.e.

$$\Delta 0_i = \sum_j \frac{\partial 0_i}{\partial X_j} \Delta X_j \ , \qquad (2.1)$$

where the X_j are the six orbit parameters. Where there are m observations available to define the six parameters, a set of m differential expressions may be written; in matrix form, this set is

$$(\Delta 0_i) = (\gamma_{ij}) (\Delta X_j) \ , \qquad (2.2)$$

where (γ_{ij}) is an m x 6 matrix. The (ΔX_j) is a six component vector, and the $(\Delta 0_i)$ is an m component vector.

If there are more observed quantities 0_i than parameters X_j, that is, when m >6, the system is overdetermined and the equations may be solved in the sense of least squares. The solution takes the form

$$(\Delta X_j) = [(\gamma_{ij})^T (\gamma_{ij})]^{-1} (\gamma_{ij})^T (\Delta 0_i) , \qquad (2.3)$$

where the -1 and T superscripts denote inverse matrix and transpose matrix respectively. The bracketed quantity in (2.3) is the so called least square matrix N

$$N = [(\gamma_{ij})^T (\gamma_{ij})] . \qquad (2.4)$$

Where more than one type of tracking data is employed, or where the statistical variances of the 0_i's are not the same, the least square matrix N must be constructed according to

$$N = [(\gamma_{ij})^T (P_{ip}) (\gamma_{ij})] \qquad (2.5)$$

where (P_{ip}) is the diagonal weight matrix, defined as

$$\begin{aligned} P_{ip} &= \frac{1}{\sigma^2_{0_i}} , \quad i = p \\ P_{ip} &= 0 , \quad i \neq p \end{aligned} \qquad (2.6)$$

σ_{0_i} is the standard deviation of the observation 0_i.

The inverse of the least square matrix N, called the variance covariance matrix, has the important property that the diagonal terms are the variances in the ΔX_j per unit observation error, assuming true Gaussian error distribution and complete independence between measurements. The unit observation error for each data type is introduced through the (P_{ip}) matrix. Thus the square root of the diagonal elements of the N^{-1} matrix is interpreted as the standard deviation in the orbit parameters arising from the selected observation pattern.

2.2 MODEL OF OBSERVATIONS NOISE CORRELATION

Once enough observations are obtained to define the geometry reasonably well, the variance in the orbit parameters decreases in manner inversely proportional to the total number of observations, when additional observations are taken over the same arc of the orbit. Or, in other terms, the orbit parameter errors decrease with the square root of the instrumentation sampling frequency.

There are, nevertheless, two regions in the sampling rate band where the square root law breaks down. The first is at very low frequency, where the total number of observations over the orbit arc are insufficient to define the geometry in detail and, as a consequence, the error in the orbit parameters decreases more rapidly than the square root law, as more observations are taken.

The second region is located where the sampling rate is significantly greater than the cutoff frequency of the observation noise and the correlation between successive data points is no longer negligible. When this happens, the gain in information obtained by increasing the sampling rate is sharply reduced. In this region the error in the orbit parameters decreases at first slower than the square root of the sampling rate and then settles at a fairly constant value.

The following model of observation noise is assumed:

(a) Statistical correlation exists between observations of the same type (range, range rate, angles) but measurements of different type are uncorrelated;

(b) The autocorrelation function of the observation noise for each type of measurement is a simple exponential:

$$\Phi(n\,T) = e^{-anT} \tag{2.7}$$

where T is the sampling period.

These assumptions may seem restrictive but they permit obtaining a good insight into the nature of the problem, they are more or less satisfied in many practical applications and, furthermore, a more sophisticated model would require a detailed knowledge of the instrumentation involved.

Let us consider first the variance covariance matrix (A_i) of one particular type of observation. Under the forestated assumptions

$$A_i = \sigma_i^2 \begin{bmatrix} 1 & e^{-a_iT} & e^{-2a_iT} & \cdots & e^{-n_ia_iT} \\ e^{-a_iT}1 & 1 & e^{-a_iT} & \cdots & e^{-(n_i-1)a_iT} \\ e^{-2a_iT} & e^{-a_iT} & 1 & \cdots & e^{-(n_i-2)a_iT} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ e^{-n_ia_iT} & e^{-(n_i-1)a_iT} & e^{-(n_i-2)a_iT} & \cdots & 1 \end{bmatrix} \tag{2.8}$$

where σ_i^2 is the variance of that type of observation, a_i the noise cutoff frequency and n_i the total number of that type of observations. The total variance-covariance matrix P then can be written in matrix form

$$P = \begin{bmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots 0 & \cdots & A_m \end{bmatrix} \tag{2.9}$$

where m is the number of types of observations. The inverse of the covariance matrix is then:

$$P^{-1} = \begin{bmatrix} A_1^{-1} & 0 & 0 & \cdots & 0 \\ 0 & A_2^{-1} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & A_m^{-1} \end{bmatrix} \tag{2.10}$$

in virtue of the partition theorem of matrix theory.

The inverse matrices A_i^{-1} $(i = 1, 2 \ldots m)$ can be obtained analytically by Z-transform methods.

$$A_i^{-1} = \frac{1}{\sigma_i^2} \begin{bmatrix} \frac{1}{1-K_i^2} & \frac{-K_i}{1-K_i^2} & 0 & 0 & 0 & 0 \\ \frac{-K_i}{1-K_i^2} & \frac{1+K_i^2}{1-K_i^2} & \frac{-K_i}{1-K_i^2} & 0 & 0 & 0 \\ 0 & \frac{-K_i}{1-K_i^2} & \frac{1+K_i^2}{1-K_i^2} & \frac{-K_i}{1-K_i^2} & 0 & 0 \\ 0 & 0 & \frac{-K_i}{1-K_i^2} & \frac{1+K_i^2}{1-K_i^2} & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \frac{1+K_i^2}{1-K_i^2} & \frac{-K_i}{1-K_i^2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{1-K_i^2} \end{bmatrix} \tag{2.11}$$

where $K_i = e^{-a_i T}$.

The variance-covariance matrix of the orbit parameters is then

$$N = (\gamma_{ij})(P)^{-1}(\gamma_{ij})^T \tag{2.12}$$

Thus the variance-covariance matrix (Eq. 2.5) must be modified slightly to account for noise correlation, since now two additional elements on each side of the elements of the main diagonal of P^{-1} become different from zero.

SECTION 3

THE $\underline{U}_o$, $\underline{V}_o$ PARAMETERS

AND THEIR DIFFERENTIAL EXPRESSIONS

For satellites of all inclinations and eccentricities, including zero in either case, the following set of parameters are recommended for the two-body calculations: (see Figure 3.1):

a - semi-major axis

$\underline{U}_o$ - unit vector directed to object from geocenter, at epoch.

$\underline{V}_o$ - unit vector normal to $\underline{U}_o$, lying in the orbit plane and in the direction of increasing anomaly.

$$\left.\begin{array}{l} e \sin E_o \\ \\ e \cos E_o \end{array}\right\} \text{ where e denotes the eccentricity, and } E_o \text{ the eccentric anomaly at epoch.}$$

These parameters avoid the singularities normally associated with elements including argument of perigee and node, at zero eccentricity and inclination respectively. Thus their use in describing the differential properties of low eccentricity, equatorial satellites, such as those proposed for communication relays, does not lead to computational difficulties. Note that the apparent seven quantities required to specify these parameters is reduced to the conventional six by the normality of $\underline{U}_o$ and $\underline{V}_o$.

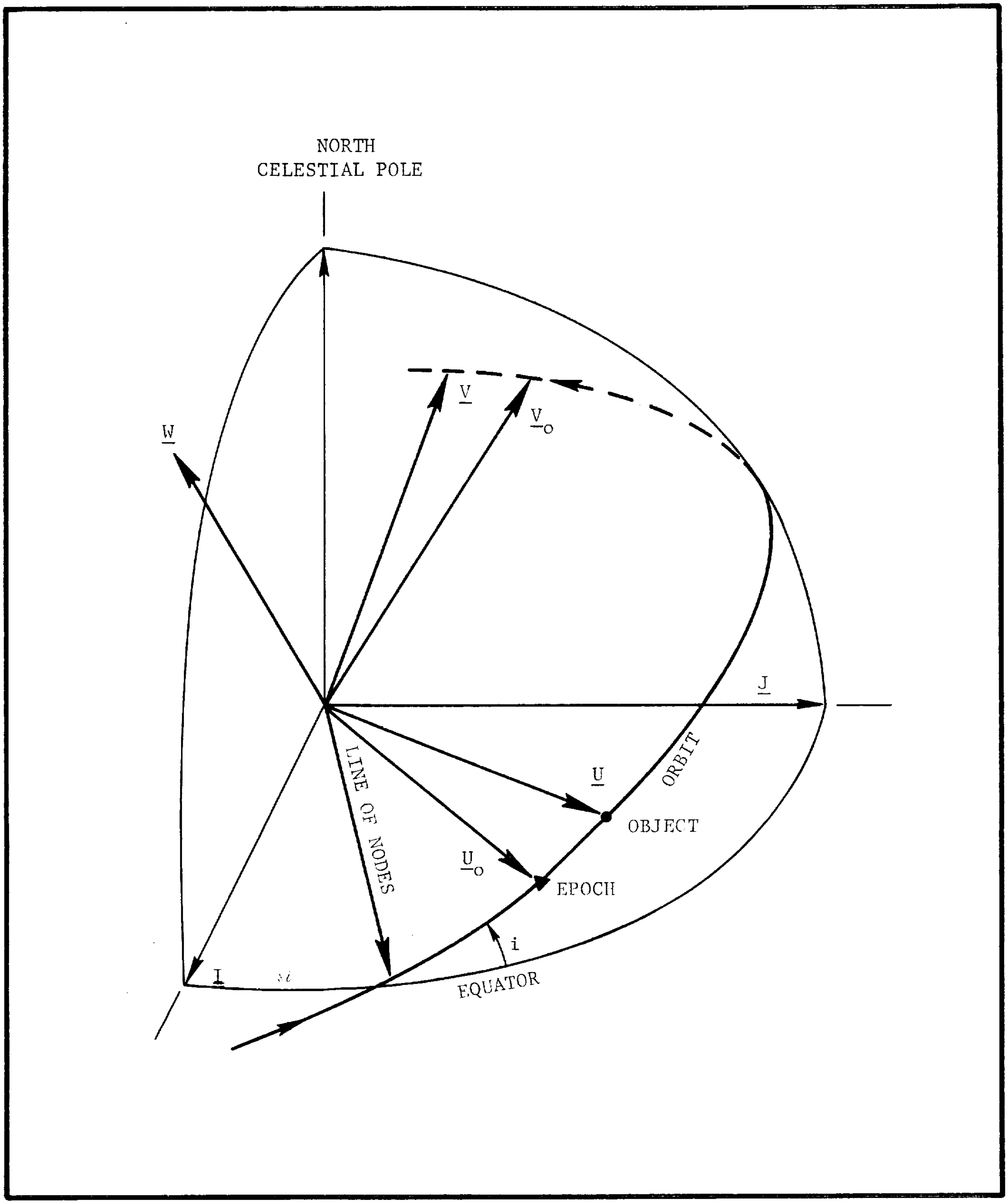

FIGURE 3.1 PROJECTION OF ORBIT ON CELESTIAL SPHERE, WITH ORIENTATION UNIT VECTORS AND ANGLES

3.1 TRANSFORMATION EQUATIONS TO INITIAL POSITION AND VELOCITY

Initial position $\underline{r}_o$ and velocity $\dot{\underline{r}}_o$ are very closely associated with these parameters, by the following formulae:

$$\underline{r}_o = r_o \underline{U}_o, \text{ where } r_o = a\,(1 - e \cos E_o)$$

$$\dot{\underline{r}}_o = \dot{r}_o \underline{U} + r \dot{v}_o \underline{V}_o, \text{ where}$$

$$\dot{r}_o = \frac{\sqrt{\mu a}}{r_o} \; e \sin E_o$$

$$r\dot{v}_o = \frac{\sqrt{\mu a \, (1 - e^2)}}{r_o}$$

The inverse transformation may be made with equal facility.

At a later (or earlier) time, the position and velocity may be derived according to the following pattern. Note that these equations have been developed to associate the coefficient e with quantities which are indeterminate for circular motion, e.g. M, E, v and ω. The quantities $(M - M_o)$, $(E - E_o)$, $(v - v_o)$, moreover, are well determined for circular motion.

Given t and the parameters a, $\underline{U}_o$, $\underline{V}_o$, $e \sin E_o$, and $e \cos E_o$ at time t_o, then:

$$M - M_o = n\,(t - t_o), \text{ where } \; n = k' \sqrt{\mu} a^{-3/2}$$

For geocentric motion, k' = 0.074, 365, 74 and the mass function μ is normally unity. The quantity μ may be augmented where perturbing bodies such as the moon affect the period of the satellites. Kepler's equation is solved in the form:

$$(E - E_o) = (M - M_o) + e \cos E_o \sin (E - E_o)$$
$$- e \sin E_o \left[1 - \cos (E - E_o)\right]$$

Next e cos E and e sin E are developed:

$$e \cos E = e \cos E_o \cos (E-E_o) - e \sin E_o \sin (E-E_o)$$

$$e \sin E = e \cos E_o \sin (E-E_o) + e \sin E_o \cos (E-E_o)$$

and

$$e^2 = (e \cos E_o)^2 + (e \sin E_o)^2$$

$$p = a (1-e^2)$$

$$r = a (1-e \cos E)$$

$$r_o = a (1-e \cos E_o)$$

and $(v-v_o)$ is derived from

$$\cos (v-v_o) = 1 - \frac{ap}{rr_o} [1 - \cos (E-E_o)]$$

$$\sin (v-v_o) = \frac{a\sqrt{ap}}{rr_o} [(M-M_o) - (E-E_o) + \sin (E-E_o)]$$

Finally

$$\underline{U} = \underline{U}_o \cos (v-v_o) + \underline{V}_o \sin (v-v_o)$$

$$\underline{V} = -\underline{U}_o \sin (v-v_o) + \underline{V}_o \cos (v-v_o)$$

and

$$\underline{r} = r\underline{U}$$

$$\dot{\underline{r}} = \dot{r}\underline{U} + r\dot{v}\ \underline{V}, \text{ where}$$

$$\dot{r} = \frac{\sqrt{\mu a}}{r}\ e \sin E$$

$$r\dot{v} = \frac{\sqrt{\mu a}}{r} \sqrt{1-e^2}$$

The inverse transformation can be developed by tracing this development backwards, beginning with $\underline{r}$ and $\dot{\underline{r}}$ at time t.

3.2 DIFFERENTIAL EXPRESSIONS

Differential expressions yield cause-and-effect type of relationships between the parameters and position or velocity, and are indispensible for either error analysis or differential correction. They are developed by differentiating the two-body representation formulae of the previous section. Only the results are presented here:

$$\begin{aligned} \Delta \underline{r} = \underline{U}\ &\left[R_a \frac{\Delta a}{a} + R_c\, \Delta(e \cos E_o) + R_s\, \Delta(e \sin E_o)\right] \\ + \underline{V}\ &\left[V_a \frac{\Delta a}{a} + V_c\, \Delta(e \cos E_o) + V_s\, \Delta(e \sin E_o) \right.\\ &\quad \left. + r\, \underline{V}_o \cdot \Delta \underline{U}_o \right] \\ + \underline{W}\ &\left[r \sin (v-v_o)\ \underline{W} \cdot \Delta \underline{V} + r \cos (v-v_o)\ \underline{W} \cdot \Delta \underline{U}_o \right] \end{aligned} \qquad (3.1)$$

The coefficients R_i and V_i are given in Table 3.1 for general eccentricity. For low eccentricity they simplify to the form:

$$R_a = a$$

$$R_c = -a \cos (v-v_o)$$

$$R_s = a \sin (v-v_o)$$

$$V_a = -\frac{3}{2} a (v-v_o)$$

$$V_c = 2a \sin (v-v_o)$$

$$V_s = -2a \left[1 - \cos (v-v_o)\right]$$

TABLE 3.1

COEFFICIENTS IN DIFFERENTIAL EXPRESSION FOR POSITION

$$R_a = r - \frac{3}{2} (M - M_o) \frac{a^2}{r} e \sin E$$

$$R_c = \frac{a^2}{r} [1 - \cos (E - E_o) - \frac{r_o}{a}]$$

$$R_s = \frac{a^2}{r} [(M - M_o) - (E - E_o) + \sin (E - E_o)]$$

$$V_a = - \frac{3}{2} (M - M_o) \frac{a^2}{r} \sqrt{1 - e^2}$$

$$V_c = \frac{a^2}{r \sqrt{1 - e^2}} \left\{ \frac{p+r}{a} \sin (E - E_o) + \frac{r}{a} (1 - \frac{r}{r_o}) e \sin E_o \right\}$$

$$V_s = \frac{a^2}{r \sqrt{1 - e^2}} \left\{ - \frac{p+r}{a} [1 - \cos (E - E_o)] + \frac{r}{a} (1 - \frac{r}{r_o}) (1 - e \cos E_o) \right\}$$

The differential expression for velocity takes the form:

$$
\begin{aligned}
\Delta \underline{\dot{r}} = \underline{U}\ & \Big[(\dot{R}_a - \dot{v} V_a) \frac{\Delta a}{a} + (\dot{R}_c - \dot{v}\, V_c)\, \Delta\, (e \cos E_o) \\
& + (\dot{R}_s - \dot{v}\, V_s)\, \Delta\, (e \sin E_o) - r\, \dot{v}\, \underline{V}_o \cdot \Delta \underline{U}_o \Big] \\
& + \underline{V}\ \Big[(\dot{V}_a + \frac{\dot{r}}{r}\, V_a) \frac{\Delta a}{a} + (\dot{V}_c + \frac{\dot{r}}{r}\, V_c)\, \Delta (e \cos E_o) \\
& + (\dot{V}_s + \frac{\dot{r}}{r} V_s)\, \Delta (e \sin E_o) + \dot{r}\, \underline{V}_o \cdot \Delta \underline{U}_o \Big] \\
& + \underline{W} \Big\{ [r\, \dot{v}\ \cos\, (v - v_o) + \dot{r}\, \sin\, (v - v_o)]\ \underline{W} \cdot \Delta \underline{V}_o \\
& - [r\, \dot{v}\, \sin\, (v - v_o) - \dot{r}\, \cos\, (v - v_o)]\ \underline{W} \cdot \Delta \underline{U}_o \Big\}
\end{aligned}
\qquad (3.2)
$$

Coefficients for arbitrary (<1) eccentricity appear in Table 3.2. For nearly circular motion, these simplify to

$$\dot{R}_a = 0$$

$$\dot{R}_c = \dot{s} \sin (v-v_o)$$

$$\dot{R}_s = \dot{s} \cos (v-v_o)$$

$$\dot{V}_a = -\frac{1}{2} \dot{s}$$

$$\dot{V}_c = \dot{s} \cos (v-v_o)$$

$$\dot{V}_s = -\dot{s} \sin (v-v_o)$$

where $\dot{s}$ is the tangential speed in the orbit.

Table 3.3 summarizes the differential expressions for position and velocity, specialized to low eccentricity. The results are expessed in local radial ($\underline{U}$ component), tangential ($\underline{V}$ component) and normal ($\underline{W}$ component) form, thus facilitating their interpretation.

TABLE 3.2

COEFFICIENTS IN DIFFERENTIAL EXPRESSION FOR VELOCITY

$$\dot{R}_a = -\frac{1}{2}\dot{r} - \frac{3}{2}(M - M_o)\sqrt{\frac{\mu}{a}}\left(\frac{a}{r}\right)^3 e(\cos E - e)$$

$$\dot{R}_c = \sqrt{\frac{\mu}{a}}\left(\frac{a}{r}\right)^3 \left\{\frac{p}{a}\sin(E - E_o) + \frac{r}{a} e \sin E_o\right\}$$

$$\dot{R}_s = \sqrt{\frac{\mu}{a}}\left(\frac{a}{r}\right)^3 \left\{-\frac{p}{a}[1 - \cos(E - E_o)] + \frac{r}{a}(1 - e \cos E_o)\right\}$$

$$\dot{V}_a = -\frac{1}{2} r \dot{v} + \frac{3}{2}(M - M_o)\sqrt{\frac{\mu}{a}}\left(\frac{a}{r}\right)^3 \sqrt{1 - e^2}\, e \sin E$$

$$\dot{V}_c = \sqrt{\frac{\mu}{a}}\left(\frac{a}{r}\right)^3 \sqrt{1 - e^2}\left[\cos(E - E_o) - e \cos E_o \left(1 - \frac{r^2}{ap}\right)\right]$$

$$\dot{V}_s = \sqrt{\frac{\mu}{a}}\left(\frac{a}{r}\right)^3 \sqrt{1 - e^2}\left[-\sin(E - E_o) + e \sin E - e \sin E_o \left(1 - \frac{r^2}{ap}\right)\right]$$

TABLE 3.3

DIFFERENTIAL EXPRESSIONS FOR $\underline{r}$ and $\underline{\dot{r}}$

SPECIALIZED TO LOW ECCENTRICITY

$$
\begin{aligned}
\frac{\Delta \underline{r}}{a} = \underline{U} \; & \left[\frac{\Delta a}{a} - \cos (v-v_o) \; \Delta (e \cos E_o) + \sin (v-v_o) \; \Delta (e \sin E_o) \right] \\
& + \underline{V} \left\{ - \frac{3}{2} \; (v-v_o) \; \frac{\Delta a}{a} + 2 \sin (v-v_o) \; \Delta (e \cos E_o) \right. \\
& \qquad \left. - 2 \left[1 - \cos (v-v_o) \right] \Delta (e \sin E_o) + \underline{V}_o \cdot \Delta \underline{U}_o \right\} \\
& + \underline{W} \left[\sin (v-v_o) \; \underline{W} \cdot \Delta \underline{V}_o + \cos (v-v_o) \; \underline{W} \cdot \Delta \underline{U}_o \right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\Delta \underline{\dot{r}}}{\dot{s}} = \underline{U} & \left\{ \frac{3}{2} \; (v-v_o) \; \frac{\Delta a}{a} - \sin (v-v_o) \; \Delta (e \cos E_o) \right. \\
& \qquad \left. + \left[2 - \cos (v-v_o) \right] \Delta (e \sin E_o) - \underline{V}_o \cdot \Delta \underline{U}_o \right\} \\
& + \underline{V} \left\{ - \frac{1}{2} \; \frac{\Delta a}{a} + \cos (v-v_o) \; \Delta (e \cos E_o) \right. \\
& \qquad \left. - \sin (v-v_o) \; \Delta (e \sin E_o) \right\} \\
& + \underline{W} \left\{ \cos (v-v_o) \; \underline{W} \cdot \Delta \underline{V}_o - \sin (v-v_o) \; \underline{W} \cdot \Delta \underline{U}_o \right\}
\end{aligned}
$$

3.3 DEVELOPMENT OF SCALAR DIFFERENTIAL EXPRESSIONS

The scalar differential expressions are developed here for range, range rate, right ascension, and declination observations. The vector differential expressions are given in Table 3.3, while the vector relationships between geocenter, observer and satellite are shown in Figure 3.2. In this analysis, the station coordinates are introduced through the $\underline{R} \cdot \underline{U}$, $\underline{R} \cdot \underline{V}$, and $\underline{R} \cdot \underline{W}$; the diurnal rotation of both earth and satellite make these quantities constant in first order analyses.

Where range measurements are available, the determination of the orbit parameters is governed by the differential relationship

$$\rho \Delta \rho = \underline{\rho} \cdot \Delta \underline{\rho} = (r\underline{U} + \underline{R}) \cdot \Delta \underline{\rho}$$

Introducing the expression for $\Delta \underline{\rho}$ (or, equivalently, $\Delta \underline{r}$ since no station errors $\Delta \underline{R}$ are considered in this present analysis) from Table 3.3, this expression is given in Table 3.4. The station location enters through scalar products with the orbit orientation vectors $\underline{U}$, $\underline{V}$, $\underline{W}$. Since the strength of the parameter determination depends upon the magnitude of the coefficients, this relationship reveals how station location enters into the determination. For example, an equatorial station leads to a zero value for $\underline{R} \cdot \underline{W}$, and consequently the orientation information conveyed in $a\underline{W} \cdot \Delta \underline{U}_o$ and $a\underline{W} \cdot \Delta \underline{V}_o$ cannot be determined from range measurements on a nearly equatorial satellite taken from equatorial sites.

The slant range rate differential expression may be developed from $\rho \dot{\rho} = \underline{\rho} \cdot \dot{\underline{\rho}}$; differentiating leads to

$$\rho \Delta \dot{\rho} = \underline{\rho} \cdot \Delta \dot{\underline{\rho}} + (\dot{\underline{\rho}} - \dot{\rho} \underline{L}) \cdot \Delta \underline{\rho}$$

Substitution of $\Delta \underline{\rho}$ and $\Delta \dot{\underline{\rho}}$ from Table 3.3, will lead to the desired expressions. Alternatively, Table 3.4 may be time differentiated

$$\rho \Delta \dot{\rho} = \frac{d}{dt} (\rho \Delta \rho) - \dot{\rho} \Delta \rho$$

noting that, for the nearly circular orbit, $\dot{v} = n$, $\dot{\underline{U}} = n\underline{V}$, etc. In addition, for the nominal equatorial and circular satellite, $\dot{\rho}$ is a differential quantity and $\dot{\rho} \Delta \rho$ leads to terms of second order; these are ignored at this state of the analysis. Either route leads to the expression given in Table 3.5.

Right ascension, α, and declination, δ, differential expressions are developed from

$$\rho\Delta\alpha\cos\delta = \underline{A}\cdot\Delta\underline{\rho} \qquad \text{and} \qquad \rho\Delta\delta = \underline{D}\cdot\Delta\underline{\rho}$$

where the $\underline{A}$ and $\underline{D}$ unit vectors are tangent to the celestial sphere at the point towards which $\underline{\rho}$ is directed. $\underline{A}$ is parallel to the equator and $\underline{D}$ is tangent to the celestial meridian, directed north, as shown in Figure 3.3. $\underline{L}$, $\underline{A}$ and $\underline{D}$ form a right handed system.

The scalar products of $\underline{A}$ and $\underline{D}$ with the $\underline{U}$, $\underline{V}$, $\underline{W}$ vectors appearing in $\Delta\underline{\rho}$ may be expressed in terms of $\underline{R}$ through the following relationships, valid for the nominal equatorial satellite:

$$\rho\cos\delta\ \underline{A}\cdot\underline{U} = -\underline{R}\cdot\underline{V}$$

$$\rho\cos\delta\ \underline{A}\cdot\underline{V} = a + \underline{R}\cdot\underline{U}$$

$$\underline{A}\cdot\underline{W} = 0$$

$$\rho\operatorname{ctn}\delta\ \underline{D}\cdot\underline{U} = -(a + \underline{R}\cdot\underline{U})$$

$$\rho\operatorname{ctn}\delta\ \underline{D}\cdot\underline{V} = -\underline{R}\cdot\underline{V}$$

$$\rho\operatorname{ctn}\delta\ \underline{D}\cdot\underline{W} = \rho\csc\delta - \underline{R}\cdot\underline{W}$$

Introducing these expressions into $\underline{A}\cdot\Delta\underline{\rho}$ and $\underline{D}\cdot\Delta\underline{\rho}$ leads to the scalar differential expressions given in Table 3.6.

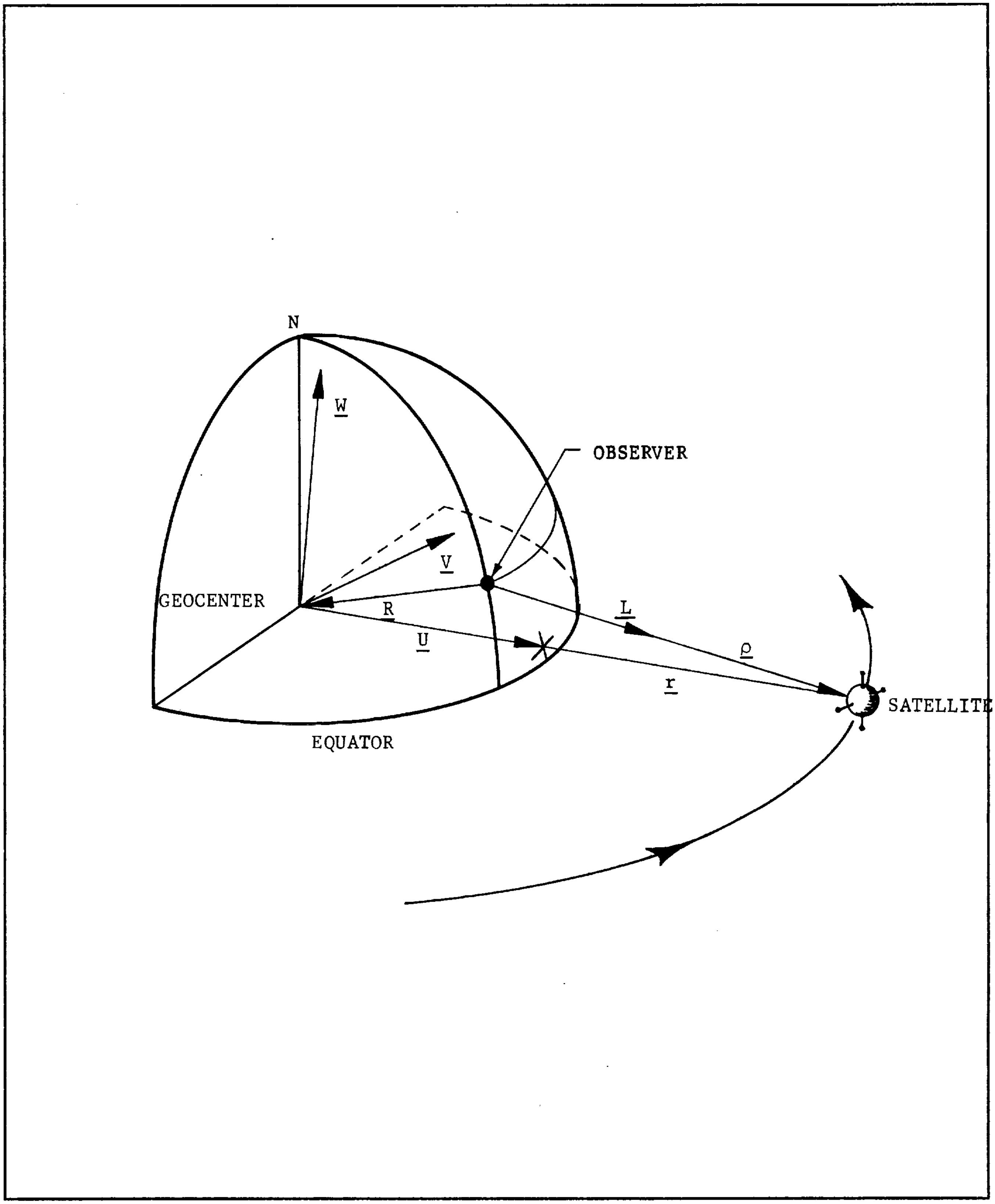

FIGURE 3.2 VECTOR RELATIONSHIPS BETWEEN SATELLITE, OBSERVER, AND GEOCENTER

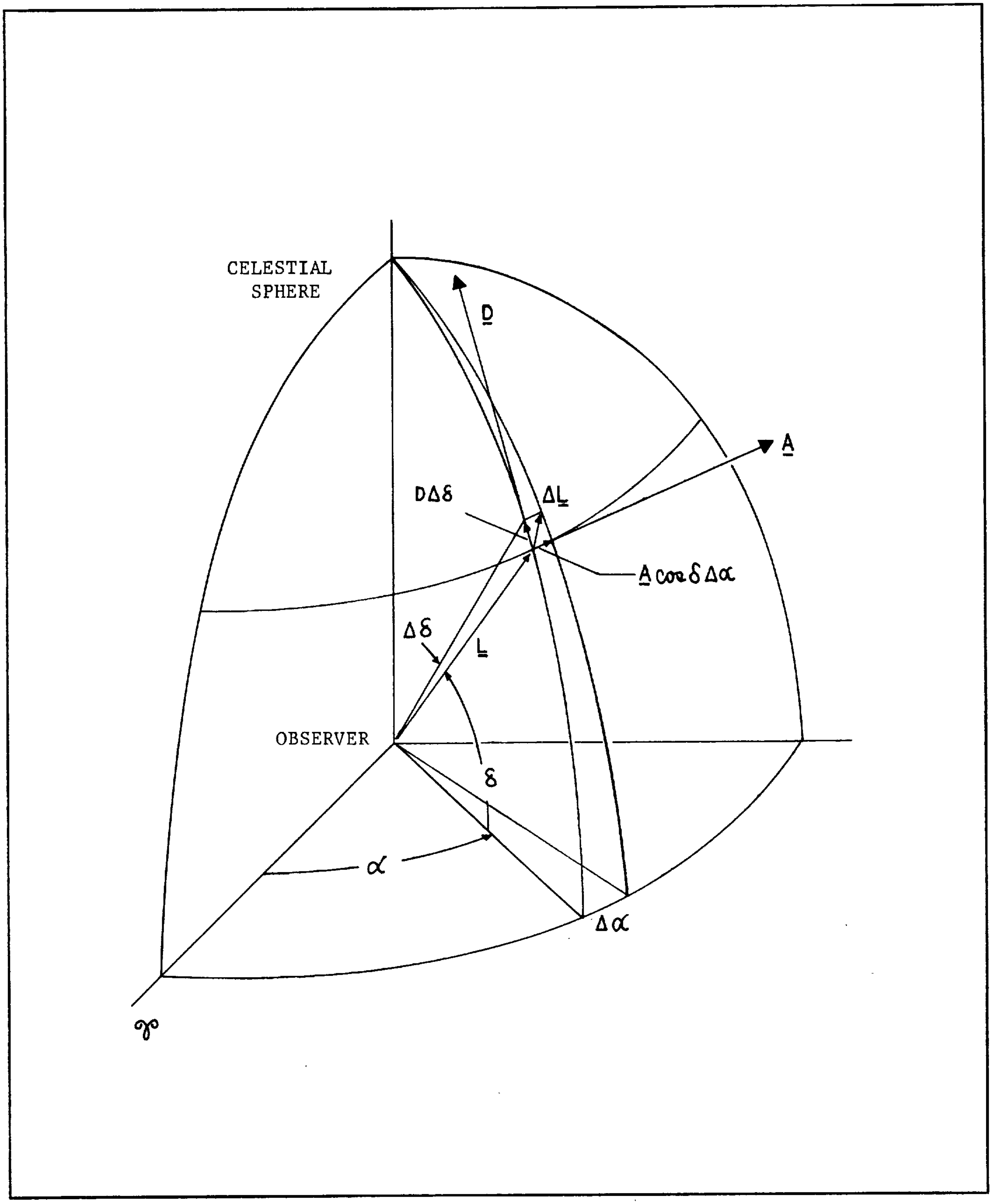

FIGURE 3.3 L, A, D VECTOR TRIAD

TABLE 3.4

SCALAR DIFFERENTIAL EXPRESSION FOR SLANT RANGE

$$\rho \Delta \rho = \left[(a + \underline{R} \cdot \underline{U}) - \frac{3}{2} (v - v_o) \underline{R} \cdot \underline{V} \right] \Delta a$$

$$+ \left[- (a + \underline{R} \cdot \underline{U}) \cos (v - v_o) + 2 \underline{R} \cdot \underline{V} \sin (v - v_o) \right] a \Delta (e \cos E_o)$$

$$+ \left[(a + \underline{R} \cdot \underline{U}) \sin (v - v_o) - 2 \underline{R} \cdot \underline{V} (1 - \cos (v - v_o)) \right] a \Delta (e \sin E_o)$$

$$+ \left[\underline{R} \cdot \underline{V} \right] a \underline{V}_o \cdot \Delta \underline{U}_o$$

$$+ \left[\underline{R} \cdot \underline{W} \sin (v - v_o) \right] a \underline{W} \cdot \Delta \underline{V}_o$$

$$+ \left[\underline{R} \cdot \underline{W} \cos (v - v_o) \right] a \underline{W} \cdot \Delta \underline{U}_o$$

TABLE 3.5

SCALAR DIFFERENTIAL EXPRESSION FOR SLANT RANGE RATE

$$\frac{\rho\Delta\dot{\rho}}{n} = \left[-\frac{3}{2}\ \underline{R}\cdot\underline{V}\right] \quad \Delta a$$

$$+\left[(a + \underline{R}\cdot\underline{U}) \sin (v-v_o) + 2\underline{R}\cdot\underline{V} \cos (v-v_o)\right] \quad a\,\Delta(e \cos E_o)$$

$$+\left[(a + \underline{R}\cdot\underline{U}) \cos (v-v_o) - 2\,\underline{R}\cdot\underline{V} \sin (v-v_o)\right] a\,\Delta(e \sin E_o)$$

$$+ \quad [0] \quad a\underline{V}_o \cdot \ \Delta\underline{U}_o$$

$$+ \quad \left[\underline{R}\cdot\underline{W} \cos (v-v_o)\right] \quad a\underline{W}\cdot \ \Delta\underline{V}_o$$

$$+\left[-\underline{R}\cdot\underline{W} \sin (v-v_o)\right] a\underline{W}\cdot\ \Delta\underline{U}_o$$

TABLE 3.6

SCALAR DIFFERENTIAL EXPRESSIONS FOR
RIGHT ASCENSION AND DECLINATION MEASUREMENTS

$$(\rho \cos \delta)^2 \Delta\alpha = \left[-\underline{R}\cdot\underline{V} - \frac{3}{2} (v-v_o) (a + \underline{R}\cdot\underline{U}) \right] \Delta a$$

$$+ \left[\cos (v-v_o) \underline{R}\cdot\underline{V} - 2 \sin (v-v_o) (a + \underline{R}\cdot\underline{U}) \right] a\Delta(e \cos E_o)$$

$$+ \left[-\sin (v-v_o) \underline{R}\cdot\underline{V} - 2 (1-\cos (v-v_o)) (a + \underline{R}\cdot\underline{U}) \right] a\Delta(e \sin E_o)$$

$$+ \left[a + \underline{R}\cdot\underline{U} \right] a\underline{V}_o \cdot \Delta\underline{U}_o$$

$$\rho^2 \operatorname{ctn} \delta \Delta \delta = \left[- (a + \underline{R}\cdot\underline{U}) + \frac{3}{2} (v-v_o) \underline{R}\cdot\underline{V} \right] \Delta a$$

$$+ \left[\cos (v-v_o) (a + \underline{R}\cdot\underline{U}) - 2 \sin (v-v_o) \underline{R}\cdot\underline{V} \right] a\Delta(e \cos E_o)$$

$$+ \left[- \sin (v-v_o) (a + \underline{R}\cdot\underline{U}) + 2 (1-\cos (v-v_o)) \underline{R}\cdot\underline{V} \right] a\Delta(e \sin E_o)$$

$$+ \left[-\underline{R}\cdot\underline{V} \right] a\underline{V}_o \cdot \Delta\underline{U}_o$$

$$+ \left[\sin (v-v_o) (\rho \csc \delta - \underline{R}\cdot\underline{W}) \right] a\underline{W} \cdot \Delta\underline{V}_o$$

$$+ \left[\cos (v-v_o) (\rho \csc \delta - \underline{R}\cdot\underline{W}) \right] a\underline{W} \cdot \Delta\underline{U}_o$$

SECTION 4

MISSION PARAMETERS

The parameters selected for the geostationary satellite tracking analysis were derived from the $\underline{U}_o$, $\underline{V}_o$ orbit parameters which are especially suited for zero inclination and zero eccentricity orbits. The mission parameters, chosen on the basis of their ease of interpretation with respect to tracking accuracy, are:

North-south excursion, which reflects orbital inclination errors,

East-west excursion, which reflects orbital eccentricity errors,

East-west bias, which reflects errors in initial positioning in the orbit, and

Time to drift out of a 2^o cone, which reflects errors in the semi-major axis.

The uncertainty in the position of the satellite's position $\underline{r}$ in terms of the $\underline{U}_o$, $\underline{V}_o$ parameters has been established in Section 3, Equation (3.1). The unit vectors $\underline{U}$, $\underline{V}$, $\underline{W}$ are so defined that $\underline{U}$ lies in the equator plane and is directed from the geocenter to the satellite, $\underline{V}$, lies in the orbit plane, is perpendicular to $\underline{U}$ and is directed from the geocenter in the direction of motion, and $\underline{W}$ completes the right hand system and is, therefore, perpendicular to the orbit plane. In the case of the geostationary satellite $\underline{W}$ is directed toward the north pole, thus the coefficient of $\underline{W}$ in equation (3.1) describes the north-south excursions:

$$\Delta_{NS} = r \sin (v-v_o)\ \underline{W} \cdot \Delta \underline{V}_o + r \cos (v-v_o)\ \underline{W} \cdot \Delta \underline{U}_o \qquad (4.1)$$

The coefficients of the vector $\underline{V}$ embody the drift as well as the EW excursions. The secular drift away from the design satellite point can be traced to uncertainty in the period of the orbit and can, therefore, be expressed in terms of uncertainty in the semi-major axis.

$$\Delta_D = -\frac{3}{2}(M - M_o)\frac{a}{r}\sqrt{1 - e^2}\;\Delta a = \frac{1}{a}V_a\;\Delta a \tag{4.2}$$

The EW excursions can be expressed as:

$$\Delta_{EW} = V_c\;\Delta(e\cos E_o) + V_s\;\Delta(e\sin E_o)\quad . \tag{4.3}$$

The remaining term in the $\underline{V}$ coefficient, $r\underline{V}_o \cdot \Delta\underline{U}_o$, represents a bias in the east west direction, thus indicating an error in the original positioning of the satellite.

Equations (4.1), (4.2), and (4.3) are of the form

$$\Delta F = \frac{\partial F}{\partial A}\;\Delta A + \frac{\partial F}{\partial B}\;\Delta B, \tag{4.4}$$

where equation (4.2) involves only one variable.

The variance of F can be computed in terms of the variances and covariances of A and B in the familiar fashion:

$$\sigma_F{}^2 = \left(\frac{\partial F}{\partial A}\right)^2\sigma_A{}^2 + \left(\frac{\partial F}{\partial B}\right)^2\sigma_B{}^2 + 2\left(\frac{\partial F}{\partial A}\right)\left(\frac{\partial F}{\partial B}\right)\sigma_{A,B}{}^2 \tag{4.5}$$

where the partials of F with respect to A and B are evaluated from initial conditions or computed at the point of interest. Similarly, equations (4.1), (4.2) and (4.3) can be expressed as:

$$\sigma_{NS}{}^2 = r^2\sin^2(v-v_o)\,\sigma^2_{\underline{W}\cdot\Delta\underline{V}_o,} + r^2\cos^2(v-v_o)\,\sigma^2_{\underline{W}\cdot\Delta\underline{U}_o} \tag{4.6}$$

$$+ 2r^2\sin(v-v_o)\cos(v-v_o)\,\sigma^2_{\underline{W}\cdot\Delta\underline{V}_o,\;\underline{W}\cdot\Delta\underline{U}_o}$$

$$\sigma_D^2 = 9/4\ (M - M_o)^2 \quad \sigma^2_{\Delta a} \tag{4.7}$$

$$\sigma_{EW}^{\ 2} = V_c^2 \quad \sigma^2_{\Delta(e \cos E_o)} + V_s^2 \quad \sigma^2_{\Delta(e \sin E_o)}$$

$$+ \ 2\, V_c V_s \quad \sigma^2_{\Delta(e \cos E_o),\ \Delta(e \sin E_o)} \tag{4.8}$$

Since the partial derivatives of the EW and NS excursions with respect to the mission parameter are periodic functions, it has been decided that the time average of these functions be used, for example, $\int_o^{T = 2} \sin^2 (v-v_o)\ dv = \frac{1}{2}$ and $\int_o^{T = 2} \sin\ (v-v_o) \cos\ (v-v_o)\ dv = 0.$ Employing this averaging process equations (4.6) and (4.8 become:

$$\sigma_{NS}^{\ 2} = 1/2\ a^2 \left[\sigma^2_{\underline{W}} \cdot \Delta \underline{V}_o + \sigma^2_{\underline{W}} \cdot \Delta \underline{U}_o \right] \tag{4.9}$$

$$\sigma_{EW}^{\ 2} = 2\ a^2 \left[\sigma^2 \ {}_{\Delta(e \cos E_o)} + 3\ \sigma^2 \ {}_{\Delta(e \sin E_o)} \right] \tag{4.10}$$

The desired topocentric angles are, therefore,

$$\theta_{EW} = \tan^{-1} \frac{\sigma_{EW}}{a - 1} \tag{4.11}$$

$$\theta_{NS} = \tan^{-1} \frac{\sigma_{NS}}{a - 1} \tag{4.12}$$

The time t (days) to drift out of a topocentric cone whose angle at the earth's surface is β(radians) can be determined from (4.7)

$$\sigma_D = (a - 1)\beta = 3\pi t\, \sigma_{\Delta a} = \frac{3}{2}(v - v_o)\sigma_{\Delta a}$$

$$t = \frac{(a - 1)\beta}{3\pi\sigma_{\Delta a}} \qquad (4.13)$$

The expressions in equations (4.11), (4.12), and (4.13), along with the EW bias expression $\underline{r}\,\sigma_{\underline{V}_o \cdot \Delta \underline{U}_o}$ represent the desired mission parameters. The σ quantities in each of these are obtained through the matrix inversion described in Section 2. The partials of the observations with respect to the orbit parameters Δa, $\Delta(e \sin E_o)$, $\Delta(e \cos E_o)$, $\underline{V}_o \cdot \Delta\underline{U}_o$, $\underline{W} \cdot \Delta\underline{V}_o$, and $W \cdot \Delta\underline{U}_o$, are computed analytically from the scalar differential expressions found in Section 3.

SECTION 5

BIAS ERRORS

In addition to random errors in the observations, certain other errors will affect the future positions of a geostationary satellite. These errors are due to the uncertainties in the constants and parameters used in the model. Some constants are used directly in the equations of motion, others are associated with the observations. The velocity of propagation and station coordinates are of the latter variety. These errors in the observed quantities are biases as contrasted with the random errors in the measurements previously considered.

The present section develops the partial derivatives of the observational quantities with respect to uncertainties in several constants. These partial derivatives can then be used in two ways. One way is to determine the effect on the observations and subsequently upon the mission parameters of an unknown bias. In the case of biased constants in the equations of motion, it is possible to predict the effect on the mission parameters directly.

The second use of the partial derivatives of observations with respect to constants, is to adjoin these partials to the matrix of partials with respect to the orbit parameters. It is then possible to solve for the amounts of the bias in the same correction process used to find the errors in the orbital elements.

5.1 STATION COORDINATE ERRORS

The station vector $\underline{R}$, from the observing station to the geocenter can be written as

$$\underline{R} = - x_c \cos\theta \, \underline{I} - x_c \sin\theta \, \underline{J} - y_c \, \underline{K}$$

where $\theta = \theta_{GR} + \lambda$ is the local sidereal time, θ_{GR} is the sidereal time at Greenwich, λ is the last longitude of the station, and $\underline{I}$, $\underline{J}$, and $\underline{K}$ are unit vectors along the x, y, and z axes, respectively. The stations meridian plane coordinates x_c and y_c are

$$x_c = (C + H) \cos \phi$$

and

$$y_c = (S + H) \sin \phi$$

where

$$C \triangleq a_e \quad (1 - e^2 \sin^2 \phi)^{-1/2}$$

$$S \triangleq \quad (1 - e^2)\ C$$

H is the height of the station above the reference spheroid

$e^2 = 2f - f^2$, where f is the flattening and e the eccentricity of the adopted reference spheroid,

ϕ is the geodetic latitude of the station

a_e is the equatorial radius of the Earth.

The fundamental relationships between the positions and velocities of the observer, satellite and dynamical center are

$$\underline{\rho} = \underline{r} + \underline{R}$$

and

$$\underline{\dot{\rho}} = \underline{\dot{r}} + \underline{\dot{R}}$$

Now, considering only the effects of station location uncertainties on the orbit, $\Delta \underline{\rho}$ can be set to zero and

$$\Delta \underline{r} = - \Delta \underline{R}$$

and

$$\Delta \underline{\dot{r}} = - \Delta \underline{\dot{R}}$$

Consequently, the effects of the station location uncertainties on the orbit can be expressed as

$$
\Delta \underline{r} \left\{
\begin{aligned}
\Delta x &= \cos\theta \sin\phi \,[-\,(C + H) + e^2\, C^3 \cos^2\phi\,]\,\Delta\phi \\
&\quad + \cos\theta \cos\phi \;\; \Delta H \\
&\quad - x_c \sin\theta\Delta\lambda \\
\Delta y &= \sin\theta \sin\phi\,[-(C + H) + e^2\, C^3 \cos^2\phi\,]\,\Delta\phi \\
&\quad + \sin\theta \cos\phi\Delta H \\
&\quad + x_c \cos\theta\Delta\lambda \\
\Delta z &= \cos\phi\,[(S + H) + e^2\, SC^2 \sin^2\phi\,]\Delta\phi \\
&\quad + \sin\phi\Delta H
\end{aligned}
\right.
$$

$$
\Delta \underline{\dot{r}} \left\{
\begin{aligned}
\Delta \dot{x} &= -\,\dot{\theta} \sin\theta \sin\phi\,[-\,(C + H) + e^2 C^3 \cos^2\phi\,]\,\Delta\phi \\
&\quad - \dot{\theta} \sin\theta \cos\phi\Delta H \\
&\quad - x_c\, \dot{\theta} \cos\theta\Delta\lambda \\
\Delta \dot{y} &= \dot{\theta} \cos\theta \sin\phi\,[-\,(C + H) + e^2 C^3 \cos^2\phi\,]\Delta\phi \\
&\quad + \dot{\theta} \cos\theta \cos\phi\;\Delta H \\
&\quad - x_c\, \dot{\theta} \sin\theta\Delta\lambda \\
\Delta \dot{z} &= 0
\end{aligned}
\right.
$$

The propagation of these position and velocity uncertainties, $\Delta \underline{r}$ and $\Delta\underline{\dot{r}}$, to the adopted parameters follow by taking the dot products of the foregoing differential expressions for $\underline{r}$ and $\underline{\dot{r}}$ as expressed in terms of the adopted parameters, with $\underline{U}$, $\underline{V}$, and $\underline{W}$. Following this procedure,

$$\frac{\Delta \underline{r}}{a} \cdot \underline{W} = \sin (v-v_o) \; \underline{W} \cdot \Delta\underline{V}_o + \cos (v-v_o) \; \underline{W} \cdot \Delta\underline{U}_o$$

$$\frac{\Delta \underline{\dot{r}}}{\dot{s}} \cdot \underline{W} = \cos (v-v_o) \; \underline{W} \cdot \Delta\underline{V}_o - \sin (v-v_o) \; \underline{W} \cdot \Delta\underline{U}_o$$

from which $\underline{W} \cdot \Delta\underline{V}_o$ and $\underline{W} \cdot \Delta\underline{U}_o$ can be found. The effects on the remaining parameters, $\frac{\Delta a}{a}$, $\Delta(e \cos E_o)$, $\Delta(e \sin E_o)$ and $\underline{V}_o \cdot \Delta\underline{U}_o$ follow from

$$\frac{\Delta \underline{r}}{a} \cdot \underline{U} = \frac{\Delta a}{a} - \cos (v-v_o) \; \Delta(e \cos E_o) + \sin (v-v_o) \; \Delta(e \sin E_o)$$

$$\frac{\Delta \underline{r}}{a} \cdot \underline{V} = -\frac{3}{2} (v-v_o) \frac{\Delta a}{a} + 2 \sin (v-v_o) \; \Delta(e \cos E_o)$$

$$- 2 \left[1 - \cos (v-v_o)\right] \Delta (e \sin E_o) + \underline{V}_o \cdot \Delta \underline{U}_o$$

$$\frac{\Delta \underline{\dot{r}}}{\dot{s}} \cdot \underline{U} = \frac{3}{2} (v-v_o) \frac{\Delta a}{a} - \sin (v-v_o) \; \Delta(e \cos E_o)$$

$$+ \left[2 - \cos (v-v_o) \right] \Delta(e \sin E_o) - \underline{V}_o \cdot \Delta \underline{U}_o$$

$$\frac{\Delta \underline{\dot{r}}}{\dot{s}} \cdot \underline{V} = -\frac{1}{2} \frac{\Delta a}{a} + \cos (v-v_o) \; \Delta(e \cos E_o) - \sin (v-v_o) \; \Delta(e \sin E_o)$$

The relationships between the uncertainties in the station location and the observations may be obtained by means of the scalar products $\underline{L} \cdot \Delta\underline{\rho}$, $\underline{A} \cdot \Delta\underline{\rho}$, and $\underline{D} \cdot \Delta\underline{\rho}$, where $\Delta\underline{\rho} = \Delta\underline{R}$. These expressions will be necessary if the matrix of partials is to be extended to include the observation versus station location partials. This adjoining of matrices is described in detail in Section 6.

Forming the above indicated vector products (refer to Section 3.3 for $\underline{L}$, $\underline{A}$, and $\underline{D}$ definitions) yields the following expressions:

$$\Delta\underline{\rho} \cdot \underline{L} = \Delta\rho = (C + H)\left[L_x \sin\theta - L_y \cos\theta\right](\cos\phi\Delta\lambda_E)$$

$$+ \left\{(L_x \cos\theta + L_y \sin\theta)\ (x_c \tan\phi - e^2 C^3 \cos^2\phi \sin\phi)\right.$$

$$\left. - L_z\ (y_c \cot\phi + SC^2 e^2 \sin^2\phi \cos\phi)\right\}\Delta\phi$$

$$- \cos\phi\,(L_x \cos\theta + L_y \sin\theta + L_z \tan\phi)\ \Delta H\ ,$$

$$\Delta\underline{\rho} \cdot \underline{A} = \rho\Delta\alpha\cos\delta = (C + H)\ \left[A_x \sin\theta - A_y \cos\theta\right](\cos\phi\Delta\lambda_E)$$

$$+ \left[x_c \tan\phi - C^3 e^2 \cos^2\phi \sin\phi\right]\left[A_x \cos\theta + A_y \sin\theta\right]\Delta\phi$$

$$- \cos\phi\ \left[A_x \cos\theta + A_y \sin\theta\right]\ \Delta H$$

$$\Delta\underline{\rho} \cdot \underline{D} = \rho\Delta\delta = (C + H)\ \left[D_x \sin\theta - D_y \cos\theta\right](\cos\phi\Delta\lambda_E)$$

$$+ \left\{\left[x_c \tan\phi - C^3 e^2 \cos^2\phi \sin\phi\right]\left[D_x \cos\theta + D_y \sin\theta\right]\right.$$

$$\left. - D_z\ \left[y_c \cot\phi + SC^2 e^2 \sin^2\phi \cos\theta\right]\right\}\Delta\phi$$

$$- \cos\phi\ \left[D_x \cos\theta + D_y \sin\theta + D_z \tan\phi\right]\ \Delta H$$

The relationship between the slant range rate and the station location uncertainties requires the range rate differential expression of Section 3.3 where the differential expression for $-\Delta\underline{\dot{r}}$ is inserted for $\Delta\underline{\dot{\rho}}$ assuming the orbit is to be uncorrected. The slant range rate expression is then

$$\rho\,\Delta\dot{\rho} = \Big\{ [-x_c \tan\phi + C^3 e^2 \sin\phi \cos^2\phi]\;[-\cos\theta\,(\eta\dot{\theta} + \dot{x} - \dot{\rho}L_x)$$

$$+ \sin\theta\,(\xi\dot{\theta} - \dot{y} + \dot{\rho}L_y)]$$

$$+ [y_c \cot\phi + SC^2 e^2 \sin^2\phi\,\cos\phi][-\dot{z} + \dot{\rho}L_z]\Big\}\,\Delta\phi$$

$$+ \Big\{\cos\phi\;[-\cos\theta(\eta\dot{\theta} + \dot{x} - \dot{\rho}L_x) + \sin\theta(\xi\dot{\theta} - \dot{y} + \dot{\rho}L_y)]$$

$$+ \sin\phi[-\dot{z} + \dot{\rho}L_{z-}]\Big\}\,\Delta H$$

$$+ (C + H)\,\Big\{\cos\theta(\xi\dot{\theta} - \dot{y} + \dot{\rho}L_y) + \sin\theta(\eta\dot{\theta} + \dot{x} - \dot{\rho}L_x)$$

$$+ x_c\dot{\theta}\Big\}\,(\cos\phi\;\Delta\lambda_E)$$

where ξ, η, ζ are the components of the topocentric vector, $\underline{\rho}$, directed from the observer to the object and $\dot{\theta}$ is the rotational rate of the Earth. The quantities $\dot{x}$ $\dot{y}$ $\dot{z}$ are the components of the vector $\underline{\dot{r}}$, representing the satellite's velocity with respect to the dynamical center. After a definitive orbit has been established, these quantities are available as a function of time.

5.2 UNCERTAINTY IN GRAVITATIONAL AND CELESTIAL CONSTANTS

The gravitational and celestial constants are used in the orbit computations to predict the motion of the satellite. This motion is controlled mainly by the gravitational field of the earth, but is perturbed by the attraction of the Sun and Moon. While it is true that the total perturbations are important, it will be shown that the uncertainty in the magnitudes of the perturbations are negligible because of the probable errors in J_2, the coefficient of the second harmonic of the geopotential, $m_☾$, the mass of the Moon, and $m_\odot$, the mass of the Sun.

There remains the uncertainty in the fundamental constant of the geopotential, which is best expressed in terms of the mass and equatorial radius of the Earth. That constant is k_e, the Gaussian gravitational constant in Earth units.

A set of elements more closely associated with the familiar a, e, i, and M_o than the $\underline{U}_o$, $\underline{V}_o$ parameters, and which still avoids the low eccentricity and low inclination singularities inherent in the former set, is used in the evaluation of effects of the uncertainties in J_2, $m_☾$, and $m_\odot$. In this set, the mean anomaly M is replaced by the mean longitude L ($L = \pi + M = \Omega + \omega + M$). Also, e and ω are replaced by e cos π and e sin π, a combination of the eccentricity and perigee location which eliminates the low eccentricity problem inherent in ω. By using as parameters the equatorial plane components of the unit vector normal to the orbit plane, that is $W_x = \sin i \sin \Omega$ and $W_y = -\sin i \cos \Omega$, a combination of i and Ω becomes available for which the secular perturbation variations reduce to zero for the geostationary satellite. The set of derived parameters applicable to a general perturbations' evaluation of the geostationary satellite orbit, is listed in Table 5.1.

TABLE 5.1

SELECTED PARAMETERS

a	semi-major axis
$W_x = \sin i \sin \Omega$	i = inclination
$W_y = -\sin i \cos \Omega$	Ω = longitude of ascending node
e cos π	e = eccentricity
e sin π	$\pi = \Omega + \omega$ = longitude of perigee
L	$L = \pi + M$ = orbital mean longitude

5.2.1 THE CONSTANT OF GRAVITATION

To evaluate the effects of the uncertainty in the gravitational constant, it is well to consider the practical method of determining and thereafter predicting the orbit.

If the orbit were determined from angular observations only, the size of the orbit in laboratory units of length would not be directly obtained, but would be inferred from the model which includes the constant under discussion. An error in k_e, the gravitational constant, which is used in both the correction and the prediction formulas, will not effect the orbital elements, if the semi-major axis is expressed in earth radii.

Even when the determination of the orbit depends heavily on slant range measurements, the period will be the most accurately determined quantity after a few days and can, therefore, be adjusted to equal 24 sidereal hours. The period is related to the semi-major axis by

$$P = \frac{2\pi a^{3/2}}{k_e\sqrt{\mu}} \left[1 - 3J_2 \left(\frac{a_e}{a}\right)^2 + 0\,(J_2^{\,2}) \right]$$

The second term shows that this mean sidereal period is decreased by the equatorial bulge by 75 parts per million or 6.5 seconds. The semi-major axis must be increased by 50 parts per million or 2.1 km to compensate.

An error in the gravitational constant will appear as a seeming systematic error in slant range because an adjustment in the semi-major axis would again compensate the period. Thus the resulting scale error is proportional to

$$\frac{\Delta a}{a} = \frac{2}{3} \quad \frac{\Delta k_e}{k_e}$$

The slant range, ρ , to the satellite from a station at a great circle distance, ϕ , from the subsatellite point is determined by

$$\rho^2 = 1 + a^2 - 2a\rho\cos\phi$$

Therefore $\quad \rho\Delta\rho = \Delta a\,(a - \cos\phi)$

and $\quad \dfrac{\partial\rho}{\partial k_e} = \dfrac{2a}{3\rho k_e}\,(a - \cos\phi)$

Under the assumption that the satellite will be corrected to have no drift, the range-rate should always be zero and, therefore, no partials of this observation with respect to variations in the constants can exist.

The distance of the observer from the center of gravity is not affected by the scale error (Figure 5.1). Thus, there will be an error in the elevation angle, h.

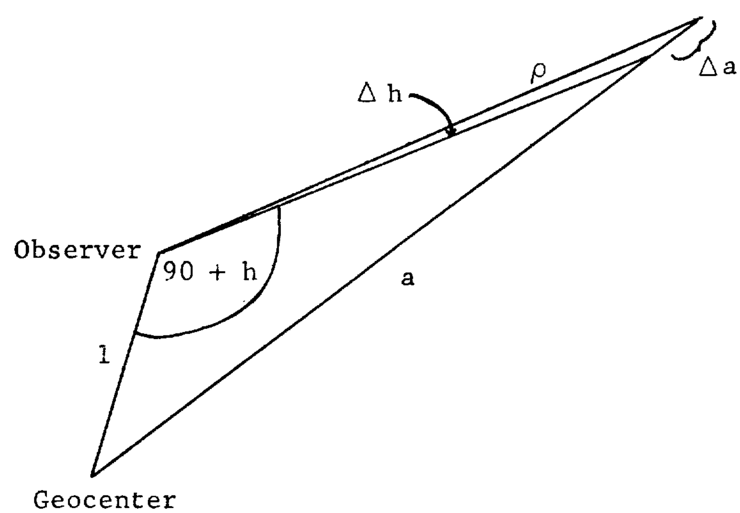

FIGURE 5.1 EFFECT OF SCALE ERROR ON ELEVATION ANGLE

Applying the law of sines to Figure 5.1

$$\frac{a}{\cos h} = \frac{1}{\cos (h+\phi)} = \frac{\rho}{\sin \phi}$$

$$a (\cos \phi \cos h - \sin \phi \sin h) = \cos h$$

$$\cos \phi - \sin \phi \tan h = \frac{1}{a}$$

From which we derive

$$\frac{a \sin\phi}{\cos^2 h} \Delta h = \frac{\Delta a}{a} = \frac{2}{3} \frac{\Delta k_e}{k_e}$$

Therefore

$$\Delta h = \frac{2}{3} \frac{\Delta k_e}{k_e} \frac{\cos h}{\rho} = \frac{2}{3} \frac{\Delta k_e}{k_e} \frac{a \sin \phi}{\rho^2}$$

or

$$\frac{\partial h}{\partial k_e} = \frac{2}{3} \frac{a \sin\phi}{k_e \rho^2}$$

which, of course, shows that the angular position with respect to the subsatellite point ($\phi = 0$) is unchanged. The elevation angle at a station 71° away is only in doubt by $\pm$ 0.1 seconds of arc because of the uncertainty in k_e (6 parts per million). This would be difficult to detect even with optical instruments.

The error in slant range, however, can be detected at nominal accuracy. The uncertainty in k_e causes an uncertainty of 150 meters in the slant range to a station 71° away.

The azimuth of the satellite is not affected by a scale error.

5.2.2 THE SECOND HARMONIC OF THE GEOPOTENTIAL

It is well known from general perturbations theory that a,e, and i have no secular variations due to J_2; Ω, ω and M do have secular variations.(1)*

Consider now the secular effects on the adopted parameters. The perturbative time variation is denoted by the grave (\`) symbol. There are no secular parts in

$$\grave{(e \cos \pi)} = - e \grave{\pi} \sin \pi + \grave{e} \cos \pi$$

and

$$\grave{(e \sin \pi)} = e \grave{\pi} \cos \pi + \grave{e} \sin \pi$$

*Numbers in parentheses denote references in Reference Section.

since the first terms have e (=0) coefficients and $\cos \pi$ and $\sin \pi$ do not combine with any of the terms in the general perturbations expansion of e to yield secular terms. Similarly, the secular portions of

$$\dot{W}_x = \dot{\Omega} \sin \dot{i} \cos \Omega + \dot{i} \cos i \sin \Omega$$

and

$$\dot{W}_y = \dot{\Omega} \sin i \sin \Omega - \dot{i} \cos i \cos \Omega$$

are zero for the geostationary satellite due to the sin i (=0) coefficients in the first terms which also appear in the expansion for i. The secular perturbation in the remaining element, L, reduces to

$$\dot{L} = 3\left(\frac{a_e}{a}\right)^2 n J_2$$

for the geostationary satellite orbit. Thus

$$\Delta \dot{L} = 3\left(\frac{a_e}{a}\right)^2 n \, \Delta J_2 .$$

A representative value for the probable error in J_2 is $\Delta J_2 = \pm 0.3 \times 10^{-6}$. Then, with a = 6.611 radii and n = 15°/hr, the uncertainty in the predicted mean longitude could be

$$\Delta \dot{L} = \pm 7\overset{\circ}{.}2 \times 10^{-6}/\text{day},$$

which represents a displacement of $\pm 0.34 \times 10^{-2}$ miles/day along the orbit. It would require 760 years for this uncertainty to amount to 2°.

This differential secular term, due to the uncertainty in J_2, acts like an uncertainty in the period and thus is indistinguishable therefrom. Moreover, none of the even harmonics can be distinguished from the fundamental in their secular effects on a geostationary orbit.

5.2.3 UNCERTAINTY IN THE MASS OF THE MOON AND THE SUN

General perturbations theory reveals that the attraction of the Moon (and Sun) on an Earth satellite will have a secular effect on only $\dot{\Omega}$, $\dot{\omega}$, and $\dot{M}$.[1],[2] For the geostationary satellite, the secular portions of $(e \cos \pi)\dot{}$, $(e \sin \pi)\dot{}$, $\dot{W}_x$, and $\dot{W}_y$ are again zero, and the secular variation in L reduces to

$$\dot{L} = -\frac{n_{☾}^2}{n} m_{☾} \left(1 - \frac{3}{2} \sin^2 i_{☾}\right).$$

The mass of the Moon, in terms of the Earth's mass, is

$$m_{☾} = \frac{1}{81.35 \pm 0.05}$$

$$= 0.012\ 2925 \pm 0.000\ 0075$$

By taking $n = 15^{\circ}/\text{hr}$, $n_{☾} = \frac{2\pi}{27.32 \text{ days}}$, and $i_{☾} = 18^{\circ}.3$, the uncertainty in $\grave{L}$ is $\pm 3.08 \times 10^{-6}$/day or a $\pm 0.14 \times 10^{-2}$ miles/day displacement along the orbit. This effect amounts to nearly half that due to the uncertainty in J_2.

This secular effect cannot be used to obtain an improvement in the lunar mass, not only because of its small size, but, again, because of confusion with the effects of the geopotential. There is some hope, however, of detecting monthly or semi-monthly variations in positions due to this source of error. The total effect on an error in the Moon's mass, however, will not significantly effect the tracking of a geostationary satellite.

The effects of an error in the Sun's mass are even smaller than those of the Moon's mass.

5.3 REFRACTION

The principal effect of atmospheric refraction on an electromagnetic wave passing through the air is a slight deviation of the ray path from a straight line. As the various portions of the wave front encounter atmospheric regions of different refractive index, their velocities will be increased or decreased relative to other portions such that the ray path bends toward regions of higher index and away from regions of lower index. In the case of missile and satellite tracking, the path of an electromagnetic signal from the vehicle generally bends downward on its way to an observing station on the ground, and as a result the apparent elevation angle is somewhat greater than would be observed in the absence of an atmosphere. Also, a range error will exist since the velocity of the wave in the atmosphere will differ from the velocity in a vacuum. Because the ray path through the atmosphere, to a target of a given altitude, is longer at low elevation angles than at high angles, refraction errors both in in angle and range will increase as the elevation angle is decreased.

5.3.1 TOTAL BENDING AND REFRACTION CORRECTIONS OF THE ELEVATION ANGLE

The basic integral formula for total ray-bending in a spherical stratified atmosphere can be accurately evaluated by approximating the refractivity profile by a series of straight-line segments, computing the bending due to each of the corresponding spherical shells, and then adding the results to obtain the total bending. The simplest sufficiently accurate formulation is that presented by Weisbrod and Anderson[3] . Let

R = mean radius of Earth

H_o = height of the station above mean sea level

H_z = height of the target above mean sea level

H_k = selected height values below ionosphere

H_j = selected height values in ionosphere

n = atmospheric index of refraction at altitude h above mean sea level

$N = (n - 1) \times 10^6$ = the refractivity

N_z = refractivity at the target height H_z

N_s = refractivity at the observation station

N_k = refractivity at H_k

N_j = refractivity at H_j

h_o = apparent elevation angle

h = true elevation angle = $h_o - \Delta h, \quad 0 \leq \Delta h$.

If the refractivity profile, i.e., the function N (h), is not known from meteorological data, it may be approximated from the world-wide charts of Bean and Horn[4].

The altitude H_z of the target above mean sea level must be known or fairly well estimated ($\pm$ 2 percent). Then:

a. Select a series of significant altitudes H_1, H_2, H_3, H_4,...., H_z, such that between each consecutive pair (H_k, H_{k+1}) the refractivity profile is well approximated by a straight line. Usually, a number of height-levels between 50 to 60 will suffice for the total atmospheric height encountered in missile and satellite work, since this will yield sufficient accuracy while at the same time retaining the effects of the relatively finer structure of the atmospheric (and in particular, ionospheric) variations.

b. For each height H_k ($k \leq L_1 \leq Z$) existing below the ionosphere, tabulate the refractivity N_k and the value tan β_k, where

$$\cos \beta_k = \frac{\cos h_o}{\left[1+H_k/(R+H_o)\right] \quad \left[1-(N_s-N_k)\times 10^{-6}\right]}$$

c. If the operating frequency does not exceed 1,000 mc, and since the target is in or above the ionosphere, then ionospheric refraction must also be considered. In this case, for each significant height H_j ($L_2 \leq j \leq Z$) in the ionosphere, tabulate the corresponding N_j and the value tan β_j where

$$\cos \beta_j = \frac{\cos h_o}{\left[1+H_j/\ (R+H_o)\right] \quad \left[1-(N_s-N_j)\times 10^{-6}\right]}$$

d. The total bending of the ray is given by

$$\gamma = \sum_{k=1}^{L_1} \frac{N_{k+1} \quad -N_k}{500(\tan \beta_{k+1} + \tan \beta_k)} + \sum_{j=l_2}^{Z} \frac{|N_{j+1}| \quad - |N_j|}{500\ (\tan \beta_{j+1} + \tan \beta_j)}$$

where the constants are chosen to yield γ in milliradians.

e. The refraction error Δh (radians) in the apparent elevation angle h_o may be computed by the formula:

$$\Delta h = \frac{10^{-3}\gamma + 10^{-6}(N_z - N_s)Y_1}{1 - (\tan h_o)\, Y_2}$$

where

$$Y_1 = \sum_{j=1}^{4} c_j\,(2_j - 1)\, X^{2j-1}$$

$$Y_2 = Y_1 + 10^{-6}\,(N_s - N_z) \sum_{j=1}^{4} c_j\,(2j-1)^2\, X^{2j-1}$$

$$X = \frac{R+H_o}{R+H_z}\cos h_o$$

and

$c_1 = 1$

$c_2 = 1/6$

$c_3 = 3/40$

$c_4 = 5/112$

The derivation of these last expressions is given in Aeronutronic Report U-954(5).

5.3.2 REFRACTION CORRECTIONS OF α, δ

The effect of refraction on α and δ follows from

$$\underline{L} = \underline{A}_o\ \Delta\alpha \cos\delta_o + \underline{D}_o\,\Delta\delta = \underline{\tilde{A}}_o\ \Delta A \cos h_o + \underline{\tilde{D}}_o\ \Delta h$$

where $\Delta A = 0$. $\underline{A}_o$ and $\underline{D}_o$ are the reference values of the $\underline{L}$, $\underline{A}$, $\underline{D}$ triad described in Section 3.3. Thus

$$\Delta\alpha = \frac{\underline{A}_o \cdot \underline{\tilde{D}}_o}{\cos \delta_o} \Delta h$$

and

$$\Delta\delta = \underline{D}_o \cdot \underline{\tilde{D}}_o \quad \Delta h$$

The apparent elevation angle h_o and azimuth A_o, required for the components of $\underline{\tilde{D}}_o$ are found from

$$\sin h_o = \sin \phi \sin \delta_o + \cos \phi \cos \delta_o \cos \alpha_o$$

$$\cos h_o \cos A_o = - \cos \phi \sin \delta_o + \sin \phi \cos \delta_o \cos \alpha_o$$

$$\sin h_o \sin A_o = \cos \delta_o \sin \alpha_o$$

Δh is computed as outlined in Section 5.3.1. The true values of α and δ are then

$$\alpha = \alpha_o + \Delta\alpha$$

and

$$\delta = \delta_o + \Delta\delta$$

5.3.3 REFRACTION CORRECTIONS FOR RANGE

The correction to the range ρ due to refraction is, in meters,

$$\Delta\rho = \sum_{k=1}^{L_1} \frac{(N_{k+1} + N_k)\ (H_{k+1} - H_k)}{1000\ (\sin \beta_{k+1} + \sin \beta_k)} + \sum_{j=L_2}^{Z} \frac{(|N_{j+1}| + |N_j|)(H_{j+1} - H_j)}{1000\ (\sin \beta_{j+1} + \sin \beta_j)}$$

The heights H are in kilometers.

Uncertainties in the elevation angle and range correction expressions for atmospheric and ionospheric refraction are presented here by means of analyses based on calculations for the 1000 mc frequency range. These calculations considered the effects of the step size used in the finite-summation approximation both through the troposphere and the ionosphere, the approximation of the shape of the ionospheric layers, and the variation in the heights of these layers. The magnitude of these individual effects are found summarized in Aeronutronic Publication U-954[5].

Figure 5.2 shows the standard deviation of angular error for the 1000 mc. frequency. Curves are shown for the low elevation angles down to 1°, which yields the greatest standard deviation.

The range uncertainty is presented in Figure 5.3 for various target heights. The maximum target height shown is 1000 n. mi., which for all practical purposes encompasses all of the refractive effects of the atmosphere and ionosphere. Thus this upper curve may be applied to the geostationary satellite orbit.

5.4 UNCERTAINTY IN THE VELOCITY OF PROPAGATION

An error in the velocity of light will be reflected in a proportional error in the slant range and range-rate observations. Since the nominal value of range-rate is zero, the error in the velocity of light will cause only a small relative error in a small discrepancy. Therefore, the only detectable effect will be on the slant range.

Since the range quantity is obtained from a time lag measurement multiplied by the velocity of propagation, that is

$$\rho = c \ (t_{final} - t_{initial})$$

then

$$\Delta\rho = \Delta c \ (t_{final} - t_{initial})$$

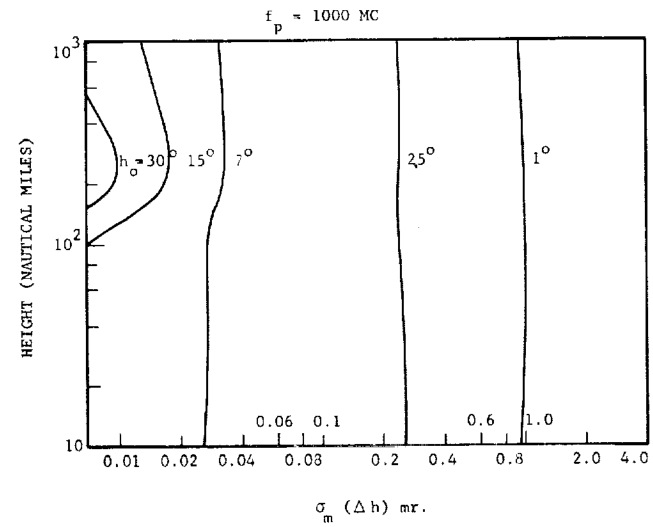

FIGURE 5.2 STANDARD DEVIATIONS, $\sigma_m(\Delta h)$, OF ANGULAR ERRORS

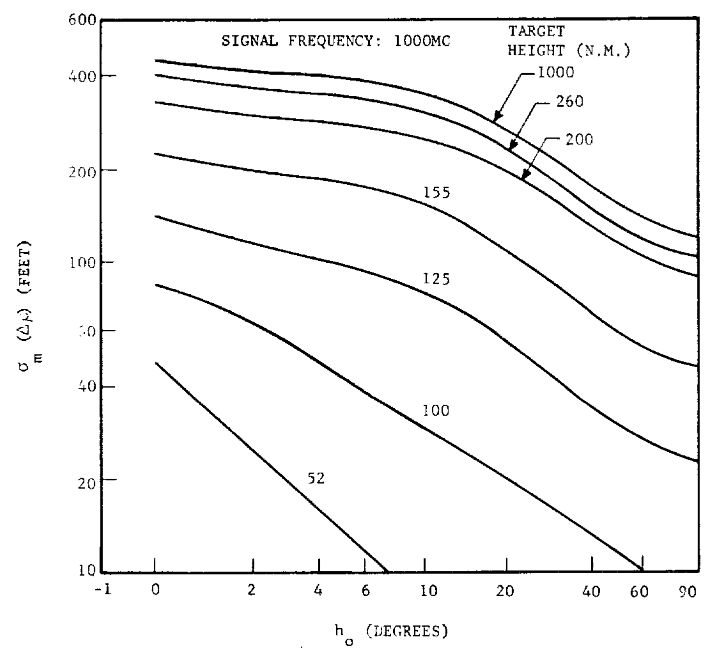

FIGURE 5.3 STANDARD DEVIATIONS, $\sigma_m(\Delta\rho)$, OF RANGE MEASUREMENTS

giving the functional relationship

$$\frac{\partial \rho}{\partial c} = \frac{\rho}{c} \quad .$$

If we assume a probable error of $\pm$ 0.3 km/sec in the velocity of propagation, there will be a corresponding probable error of $\pm$ 41 meters in the slant range from a station 71° away from the subsatellite point to the satellite. This error is always proportional to range. At the subsatellite point, the probable error in slant range is $\pm$ 36 meters.

SECTION 6

TREATMENT OF BIAS ERRORS

Generally every effort is made to correct for all predictable and measurable errors by means of precalibration of the tracking systems and by such standard corrective procedures as the corrections for velocity of propagation. Still the need exists to incorporate in the orbit determination methods means of treating systematic or bias errors in the tracking data, specifically (a) uncertainties in the tracking stations' location; (b) uncertainties in the gravitational and celestial constants used to describe the earth, the moon and the sun; (c) velocity of propagation and atmospheric refraction. The calibration of the tracking instruments is also subject to systematic errors, and biases are caused by mechanical factors, such as misalignments and deformations in the radar mounts.

6.1 EFFECT OF BIAS ERRORS ON DIFFERENTIAL CORRECTION

The differential correction procedure of orbit determination in its classical form assumes the observations' residuals to be normally distributed and unbiased. In order to estimate systematic errors, it is natural to accept the hypothesis of unbiased residuals and iteratively determine the six orbit parameters by means of the differential correction method. The residuals obtained from the last iteration are then statistically tested to reveal the presence of bias terms or low period correlation. If the hypothesis of unbias and statistical independence of the residuals is rejected at a high confidence level, one must conclude that one or more significant sources of systematic errors are present in the tracking network.

The structure of the residuals will usually indicate what sources of bias are more likely to have caused the non-random behavior of the residuals. In this first approach, relationships between various types of bias errors and orbit parameters are obtained in order to evaluate the effects of various sources of systematic errors. The magnitude of the errors can be derived from "a priori" knowledge or can be estimated from the observations' residuals.

6.2 ERROR ANALYSIS FOR A SIX PARAMETER DIFFERENTIAL CORRECTION

Let us review briefly the basic principles of the differential correction procedure. The differential relations between the 6-column vector, X ($= x_1, x_2, \ldots, x_6$) of the orbit parameters and the m-column vector, 0 ($= o_1, o_2, \ldots, o_m$) of the observations can be expressed in matrix form as

$$\Delta 0 = \Gamma \Delta X \tag{6.1}$$

where $\Gamma = [\gamma_{ij}]$ is a m x 6 matrix, whose generic element is

$$\gamma_{ij} = \frac{\partial 0_i}{\partial x_j}$$

The likelihood functional, in the case of Gaussian noise, is the weighted sum of the squares and crossproducts of the residuals:

$$(\Delta 0 - \Gamma \Delta X)^T P^{-1} (\Delta 0 - \Gamma \Delta X) = \text{Min.} \tag{6.2}$$

where P is the covariance matrix of the observations'residuals and reduces to a diagonal matrix when the observations are linearly independent. The set of normal equations which results from the minimization of the functional (6.2) can be expressed in matrix form as:

$$\Gamma^T P^{-1} \Gamma \Delta X = \Gamma^T P^{-1} \Delta 0 \tag{6.3}$$

let

$$N = \Gamma^T P^{-1} \Gamma .$$

The inverse of this matrix, N^{-1}, can be shown to be the covariance matrix of the orbit parameters' corrections

$$N^{-1} = \overline{\Delta X^T \Delta X}$$

In order to investigate the effects of bias errors on the orbit parameters, let us define as r-column vector, Y (= y_1, y_2,..., y_r) representative of all systematic errors in stations location, celestial and gravitational constants, velocity of propagation, etc. Let us also define the r x m matrix A = $[\alpha_{ij}]$, whose generic element is defined as

$$\alpha_{ij} = \frac{\partial 0_i}{\partial y_j}$$

The differential relations between observations and biases can then be expressed in matrix form as

$$\Delta 0 = A \Delta Y \tag{6.4}$$

Then, if one were to ignore the presence of biases and adopt a six parameter differential correction procedure, the following systematic errors, ΔX_b, would be introduced in the six orbit parameters by the bias terms:

$$\Delta X_b = N^{-1} \Gamma^T P^{-1} A \Delta Y \tag{6.5}$$

All the matrices appearing on the right side of this expression, with the exception of matrix A, are immediately available in the program already developed for statistical studies, as it can be noticed by comparing equations (6.5) and (6.3).

Analytical expressions relating the various types of bias errors and the observations such as those derived in Section 5.1 make up the A matrix elements.

6.3 ESTIMATION OF BIAS ERRORS BY DIFFERENTIAL CORRECTION

The differential correction method can be generalized to take into consideration biases and estimate them together with the orbit parameters. One can form the (r + 6) x m matrix Γ_1, by joining the two matrices Γ and A:

$$\Gamma_1 = |\Gamma \vdots A|$$

and indicate by ΔX_1 the column vector of the orbit parameters and bias errors

$$\Delta X_1 = \begin{vmatrix} x_1 \\ \vdots \\ x_6 \\ y_1 \\ \vdots \\ y_r \end{vmatrix}$$

Then one can write in matrix form:

$$\Delta 0 = \Gamma_1 \Delta X_1 \tag{6.6}$$

The likelihood of the sample can then be maximized both with respect to the orbit parameters' corrections and the bias errors, obtaining

$$\Delta X_1 = N_1^{-1} \Gamma_1^T P^{-1} \Delta 0 \tag{6.7}$$

where

$$N_1 = \Gamma_1^T P^{-1} \Gamma_1 \tag{6.8}$$

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